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Constructive proof of existence for a class of fourth-order nonlinear BVPs

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ABSTRACT

A new existence proof of solutions for a class of fourth-order nonlinear boundary value problems is proposed. The proof of the main results is based on the reproducing kernel theorem. It is worthwhile to point out that the method presented in this paper can be applied for the existence proof of diverse kinds of boundary conditions.

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1. Introduction

In this paper, we consider the existence of solutions for the following nonlinear fourth-order ordinary differential equation

$$u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \quad 0 < x < 1, \tag{1.1}$$

with the boundary conditions

$$u(0) = 0, \qquad u'(1) = 0, \qquad au''(0) - bu'''(0) = 0, \qquad cu''(1) + du'''(1) = 0, \tag{1.2}$$

where $a, b, c, d \geq 0$, $\rho := ad + bc + ac > 0$ and $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous.

Fourth-order BVPs play an important role in studying a large class of elastic deflection [1–3]. Therefore, a lot of studies have been devoted to the existence of solutions of Eq. (1.1).

In [4], Han and Xu considered the following fourth-order ordinary differential equation with Navier boundary conditions

$$u^{(4)}(x) = f(x, u(x)), \quad 0 < x < 1, \tag{1.3}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{1.4}$$

The authors studied the existence and multiplicity of solutions by using Morse theory.

In [5], combining truncation techniques with a variational approach, the authors established an existence result for nontrivial solutions of the following problems

$$u^{(4)}(x) + q(x)u''(x) + \alpha(x)u(x) = f(x, u(x), u'(x), u''(x)), \quad 0 < x < 1, \tag{1.5}$$

with boundary conditions (1.4). These results considered only an equation of the form

$$u^{(4)}(x) = f(x, u(x)) \quad \text{or} \quad u^{(4)}(x) = f(x, u(x), u'(x), u''(x)).$$

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In [6], the authors employed the upper and lower solutions method to study the existence of a solution for Eq. (1.1) with boundary condition

$$u(0) = u'(1) = u''(0) = u'''(1) = 0. \tag{1.6}$$

Nonlinear function f satisfied a Nagumo-type condition with respect to φ, ψ, that was, there existed a positive function $h(s)$ on $[0, \infty]$ satisfying

$$|f(x, y, z, w, v)| \leq h(|v|),$$

for all $(x, y, z, w, v) \in [0, 1] \times [-M, M]^2 \times [\varphi, \psi] \times R$, and

$$\int_{\lambda}^{\infty} \frac{s}{h(s)} \mathrm{d}s > \max_{0\leq t\leq 1} \psi(t) - \min_{0\leq t\leq 1} \varphi(t),$$

where $\alpha'' = \varphi$, $\beta'' = \psi$,α, β were lower and upper solutions to Eqs (1.1), respectively, $\lambda = \max\{|\psi(1) - \varphi(0)|, |\psi(0) - \varphi(1)|\}$, $\varphi, \psi \in C([0, 1], R)$ and $\varphi(t) - \psi(t) \leq 0, t \in [0, 1]$. It is clear that there was strong restriction on f and it was difficult to verify the condition on f.

Recently, Feng, Ji and Ge presented the existence and uniqueness of solution for BVP (1.1)–(1.2) by means of lower and upper solutions method in [7]. Nonlinearity f satisfied also the Nagumo-type condition in [6].

Motivated by the above work, in this paper, we will obtain an existence theorem of the solutions of BVPs (1.1)–(1.2) in the reproducing kernel space by a constructive method. The method has several advantages. Firstly, the conditions for determining solutions in BVPs (1.1)–(1.2) can be imposed on the reproducing kernel space and the reproducing kernel satisfying the conditions for determining solutions can be calculated. We will use the kernel to prove the existence. Therefore, the existence for diverse kinds of boundary conditions such as (1.2), (1.4) and (1.6) can be obtained. Next, the conditions on nonlinearity f are simple and it is easy to verify them. Finally, the method of obtaining the existence of the solution is constructive and the iterative sequence $u_n(x)$ converges in $C^4[0, 1]$ to the exact solution $u(x)$ of BVPs (1.1)–(1.2).

The outline of this paper is as follows. In Section 2, two definitions of reproducing kernel spaces, a linear operator and some essential lemmas are introduced. In Section 3, an iterative method for the existence of solutions for BVPs (1.1)–(1.2) based on reproducing kernel space is described. This article ends in Section 4 with some conclusions.

2. Preliminaries

In this paper, we will prove the existence for boundary condition corresponding to the case $a, b, c, d > 0$. When $abcd = 0$, the case is similar to that of boundary conditions (1.4) or (1.6).

2.1. The reproducing kernel spaces $W_5[0, 1]$ and $W_1[0, 1]$

Definition 2.1. $W_5[0, 1] = \{u(x) \mid u^{(4)}(x)$ is an absolutely continuous real value function in $[0, 1]$, $u^{(5)}(x) \in L^2[0, 1]$, $u(0) = 0$, $u'(1) = 0$, $au''(0) - bu'''(0) = 0$, $cu''(1) + du'''(1) = 0\}$. The inner product and norm in $W_5[0, 1]$ are given respectively by

$$\langle u(x), v(x)\rangle_{W_5} = u^{(4)}(0)v^{(4)}(0) + \int_0^1 u^{(5)}(x)v^{(5)}(x)\mathrm{d}x, \qquad \| u \|_{W_5} = \sqrt{\langle u, u\rangle_{W_5}}, \tag{2.1}$$

where $a, b, c, d > 0$, $\rho := ad + bc + ac > 0$.

We have to prove that $\langle u(x), v(x)\rangle_{W_5}$ satisfies all the requirements for the inner product. It is clearly seen that (i) $\langle u(x), u(x)\rangle_{W_5} \geq 0$; (ii) $\langle u(x), v(x)\rangle_{W_5} = \langle v(x), u(x)\rangle_{W_5}$; (iii) $\langle \alpha u(x), v(x)\rangle_{W_5} = \alpha\langle u(x), v(x)\rangle_{W_5}$; (iv) $\langle u(x) + w(x), v(x)\rangle_{W_5} = \langle u(x), v(x)\rangle_{W_5} + \langle w(x), v(x)\rangle_{W_5}$, where $u(x), v(x), w(x) \in W_5[0, 1]$ and $\alpha \in \mathbb{K}$.

It therefore remains only to prove that $\langle u(x), u(x)\rangle_{W_5} = 0$ only when $u(x) = 0$. In fact, it is obvious that when $u(x) = 0$, $\langle u(x), u(x)\rangle_{W_5} = 0$. On the other hand if $\langle u(x), u(x)\rangle_{W_5} = 0$, then by (2.1), we have

$$\langle u(x), u(x)\rangle_{W_5} = (u^{(4)}(0))^2 + \int_0^1 (u^{(5)}(x))^2\mathrm{d}x = 0,$$

therefore

$$u^{(4)}(0) = 0, \qquad u^{(5)}(x) = 0.$$

Then we can obtain $u^{(4)}(x) = u^{(5)}(x) = 0$ and

$$u'''(x) = \gamma, \qquad u''(x) = \gamma x + \beta, \tag{2.2}$$

where γ, β are undetermined constants.

Since $u(x) \in W_5[0, 1]$, it follows that

$$au''(0) - bu'''(0) = 0, \qquad cu''(1) + du'''(1) = 0. \tag{2.3}$$

Substituting (2.2) into (2.3), we get

$$\begin{cases} a\beta - b\gamma = 0, \\ c(\gamma+\beta) + d\gamma = 0. \end{cases}$$

By $a, b, c, d > 0$ and $ac + bc + ad > 0$, we can have $\beta = \gamma = 0$. Thus, $u'''(x) = u''(x) = 0$. Since $u(0) = u'(1) = 0$, it holds that $u(x) = u'(x) = 0$.

Theorem 2.1. *The space $W_5[0, 1]$ is a complete reproducing kernel space. That is, for each fixed $x \in [0, 1]$, there exists $K_5(y, x) \in W_5[0, 1]$, such that $\langle u(y), K_5(y, x)\rangle_{W_5} = u(x)$ for any $u(y) \in W_5[0, 1]$ and $y \in [0, 1]$. The reproducing kernel $K_5(y, x)$ can be written as*

$$K_5(y,x) = \begin{cases} \sum_{i=1}^{10} a_i y^{i-1}, & y \le x, \\ \sum_{i=1}^{10} b_i y^{i-1}, & y > x. \end{cases} \tag{2.4}$$

Proof. The proof of the completeness and reproducing property of $W_5[0, 1]$ is similar to the proof of Theorem 2.1 and Theorem 2.2 in [8].

Now, let us find out the expression of the reproducing kernel function $K_5(y, x)$ in $W_5[0, 1]$. Through several integrations by parts for (2.1), we have

$$\begin{aligned} \langle u(y), K_5(y,x)\rangle_{W_5} &= \sum_{i=0}^{3}(-1)^{3-i}u^{(i)}(0)\frac{\partial^{9-i}K_5(0,x)}{\partial y^{9-i}} + u^{(4)}(0)\left(\frac{\partial^4 K_5(0,x)}{\partial y^4} - \frac{\partial^5 K_5(0,x)}{\partial y^5}\right) \\ &\quad + \sum_{i=0}^{4}(-1)^{4-i}u^{(i)}(1)\frac{\partial^{9-i}K_5(1,x)}{\partial y^{9-i}} - \int_0^1 u(y)\frac{\partial^{10}K_5(y,x)}{\partial y^{10}}\,\mathrm{d}y. \end{aligned}$$

Since $u(x) \in W_5[0, 1]$, it follows that

$$u(0) = 0, \qquad u'(1) = 0, \qquad au''(0) - bu'''(0) = 0, \qquad cu''(1) + du'''(1) = 0,$$

then

$$\begin{aligned} \langle u(y), K_5(y,x)\rangle_{W_5} &= u'(0)\frac{\partial^8 K_5(0,x)}{\partial y^8} + u'''(0)\left(\frac{\partial^6 K_5(0,x)}{\partial y^6} - \frac{b}{a}\frac{\partial^7 K_5(0,x)}{\partial y^7}\right) + u^{(4)}(0)\left(\frac{\partial^4 K_5(0,x)}{\partial y^4} - \frac{\partial^5 K_5(0,x)}{\partial y^5}\right) \\ &\quad + u(1)\frac{\partial^9 K_5(1,x)}{\partial y^9} + u'''(1)\left(\frac{-\partial^6 K_5(1,x)}{\partial y^6} - \frac{d}{c}\frac{\partial^7 K_5(1,x)}{\partial y^7}\right) + u^{(4)}(1)\frac{\partial^5 K_5(1,x)}{\partial y^5} - \int_0^1 u(y)\frac{\partial^{10}K_5(y,x)}{\partial y^{10}}\,\mathrm{d}y. \end{aligned}$$

Note that the definition of the reproducing kernel $\langle u(y), K_5(y, x)\rangle_{W_5} = u(x)$, $K_5(y, x)$ is the solution of the following generalized differential equation

$$\frac{\partial^{10}K_5(y,x)}{\partial y^{10}} = -\delta(y-x), \tag{2.5}$$

with the boundary conditions

$$\begin{cases} \dfrac{\partial^8 K_5(0,x)}{\partial y^8} = 0, \\ \dfrac{\partial^6 K_5(0,x)}{\partial y^6} - \dfrac{b}{a}\dfrac{\partial^7 K_5(0,x)}{\partial y^7} = 0, \\ \dfrac{\partial^4 K_5(0,x)}{\partial y^4} - \dfrac{\partial^5 K_5(0,x)}{\partial y^5} = 0, \\ \dfrac{\partial^9 K_5(1,x)}{\partial y^9} = 0, \\ \dfrac{-\partial^6 K_5(1,x)}{\partial y^6} - \dfrac{d}{c}\dfrac{\partial^7 K_5(1,x)}{\partial y^7} = 0, \\ \dfrac{\partial^5 K_5(1,x)}{\partial y^5} = 0. \end{cases} \tag{2.6}$$

When $y \neq x$, it is easy to know that $K_5(y, x)$ is the solution of the following constant linear homogeneous differential equation with 10 orders

$$\frac{\partial^{10} K_5(y,x)}{\partial y^{10}} = 0,$$

therefore, $K_5(y,x)$ can be written as

$$K_5(y,x) = \begin{cases} \sum_{i=1}^{10} a_i y^{i-1}, & y \le x, \\ \sum_{i=1}^{10} b_i y^{i-1}, & y > x. \end{cases} \tag{2.7}$$

From (2.5) and the definition of $W_5[0, 1]$, we have

$$\begin{cases} \dfrac{\partial^k K_5(x+0,x)}{\partial y^k} = \dfrac{\partial^k K_5(x-0,x)}{\partial y^k}, & k = 0, 1, 2, \ldots, 8, \\ \dfrac{\partial^9 K_5(x-0,x)}{\partial y^9} - \dfrac{\partial^9 K_5(x+0,x)}{\partial y^9} = 1, \\ u(0) = 0, \\ u'(1) = 0, \\ au''(0) - bu'''(0) = 0, \\ cu''(1) + du'''(1) = 0. \end{cases} \tag{2.8}$$

According to (2.6) and (2.8), the unknown coefficients a_i and b_i $(i = 1, 2, \ldots, 10)$ in (2.7) can be solved easily, and therefore the reproducing kernel function $K_5(y,x)$ is obtained. $\square$

We collect the following two corollaries in [8] for future use.

Corollary 2.1.

$$\frac{\partial^{i+j} K_5(y,x)}{\partial x^i \partial y^j} \in L^2[0, 1], \quad i + j = 9.$$

Corollary 2.2.

$$\frac{\partial^{i+j} K_5(y,x)}{\partial x^i \partial y^j}, \quad 0 \le i + j \le 8$$

is an absolutely continuous function in [0, 1] with respect to x or y.

Definition 2.2. $W_1[0, 1] = \{u(x) \mid u(x)$ is an absolutely continuous real value function in $[0, 1]$, $u'(x) \in L^2[0, 1]\}$. The inner product and norm in $W_1[0, 1]$ are given respectively by

$$\langle u(x), v(x)\rangle_{W_1} = u(0)v(0) + \int_0^1 u'(x)v'(x)\mathrm{d}x, \qquad \| u \|_{W_1} = \sqrt{\langle u, u\rangle_{W_1}}.$$

In Ref. [8], it had been proved that $W_1[0, 1]$ is a complete reproducing kernel space and its reproducing kernel is

$$K_1(y,x) = \begin{cases} 1 + y, & y \le x, \\ 1 + x, & y > x. \end{cases}$$

2.2. Introduction to a bounded linear operator and a complete normal orthogonal system in $W_5[0, 1]$

Define a linear operator $\mathbb{L} : W_5[0, 1] \to W_1[0, 1]$,

$$\mathbb{L}u(x) = u^{(4)}(x),$$

then BVPs (1.1)–(1.2) can be converted into the equivalent form as follows

$$\begin{cases} \mathbb{L}u = f(x, u(x), u'(x), u''(x), u'''(x)), & 0 < x < 1, \\ u(0) = 0, \\ u'(1) = 0, \\ au''(0) - bu'''(0) = 0, \\ cu''(1) + du'''(1) = 0, \end{cases} \tag{2.9}$$

where $u(x) \in W_5[0, 1]$ and $f(x, u, u', u'', u''') \in W_1[0, 1]$ as $u = u(x) \in W_5[0, 1]$. It is easy to prove that $\mathbb{L}$ is a bounded linear operator.

Now, we construct an orthogonal function system.

Let $\varphi_i(x) = K_1(x, x_i)$, $\psi_i(x) = \mathbb{L}^*\varphi_i(x)$, where $\mathbb{L}^*$ is the conjugate operator of $\mathbb{L}$.

The following Lemmas 2.1 and 2.2 are collected (see [9]) for future use.

Lemma 2.1. *If $\{x_i\}_{i=1}^{\infty}$ is dense on [0, 1], then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system in $W_5[0,1]$ and $\psi_i(x) = \mathbb{L}_y K_5(y,x)|_{y=x_i}$. The subscript y by the operator $\mathbb{L}$ indicates that the operator $\mathbb{L}$ applies to the function of y.*

Lemma 2.2. *If $u(x) \in W_5[0,1]$, then there exists $M_1 > 0$, such that $\|u\|_{C^4} \le M_1\|u\|_{W_5}$, where $\|u\|_{C^4} = \max_{0\le x\le 1}\{|u(x)| + |u'(x)| + |u''(x)| + |u'''(x)| + |u^{(4)}(x)|\}$*

The normal orthogonal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ in $W_5[0,1]$ can be derived from Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x),$$

where β_{ik} are orthogonalization coefficients, $\beta_{ii} > 0$, $i = 1, 2, \ldots$.

Lemma 2.3. *Let $\{u_n(x)\}_{n=1}^{\infty}$ be a bounded set in $W_1[0,1]$, that is, there exists a $K > 0$, such that $\|u_n\|_{W_1} \le K$, then $\{u_n(x)\}_{n=1}^{\infty}$ is a compact set in $C[0,1]$*

Proof. (i) Since

$$|u_n(x)| = |\langle u_n(y), K_1(y,x)\rangle_{W_1}| \le \|u_n\|_{W_1}\sqrt{K_1(x,x)} \le K\sqrt{1+x} \le K\sqrt{1+1} = \sqrt{2}K,$$

$\{u_n(x)\}_{n=1}^{\infty}$ is uniformly bounded in $C[0,1]$.

(ii) For $\forall x_1, x_2 \in [0,1]$, by the representation of $K_1(y,x)$, we have

$$\begin{aligned}
|u_n(x_1) - u_n(x_2)| &= |\langle u_n(y), K_1(y,x_1) - K_1(y,x_2)\rangle_{W_1}| \\
&\le \|u_n\|_{W_1}\|K_1(y,x_1) - K_1(y,x_2)\|_{W_1} \\
&\le K\sqrt{K_1(x_1,x_1) + K_1(x_2,x_2) - 2K_1(x_2,x_1)} \\
&\le K\sqrt{|x_1 - x_2|}.
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, takes $\delta = \frac{\varepsilon^2}{K^2}$, we obtain $|u_n(x_1) - u_n(x_2)| < \varepsilon$ for $|x_1 - x_2| < \delta$, $\forall n \in N$, so $\{u_n(x)\}_{n=1}^{\infty}$ is equicontinuous.

From (i) and (ii), we know that $\{u_n(x)\}_{n=1}^{\infty}$ is a compact set in $C[0,1]$. □

Lemma 2.4. *If $\{u_n(x)\}_{n=1}^{\infty}$, $\{u_n'(x)\}_{n=1}^{\infty}$, $\{u_n''(x)\}_{n=1}^{\infty}$, $\{u_n'''(x)\}_{n=1}^{\infty}$ and $\{u_n^{(4)}(x)\}_{n=1}^{\infty}$ are uniformly bounded on [0, 1], that is, there exists a constant $M > 0$, such that $|u_n(x)| \le M$, $|u_n'(x)| \le M$, $|u_n''(x)| \le M$, $|u_n'''(x)| \le M$ and $|u_n^{(4)}(x)| \le M$ for any $n \in \mathbb{N}$ and $x \in [0,1]$, then $\{u_n(x)\}_{n=1}^{\infty}$, $\{u_n'(x)\}_{n=1}^{\infty}$, $\{u_n''(x)\}_{n=1}^{\infty}$ and $\{u_n'''(x)\}_{n=1}^{\infty}$ are compact sets in $C[0,1]$ respectively.*

Proof. (i) By Lemma 2.3, we only need to prove that $\{u_n(x)\}_{n=1}^{\infty}$, $\{u_n'(x)\}_{n=1}^{\infty}$, $\{u_n''(x)\}_{n=1}^{\infty}$ and $\{u_n'''(x)\}_{n=1}^{\infty}$ are bounded sets in $W_1[0,1]$ respectively.

$$\|u_n'''\|_{W_1}^2 = (u_n'''(0))^2 + \int_0^1 (u^{(4)}(x))^2 dx \le 2M^2,$$

$$\|u_n''\|_{W_1}^2 = (u_n''(0))^2 + \int_0^1 (u'''(x))^2 dx \le 2M^2,$$

$$\|u_n'\|_{W_1}^2 = (u_n'(0))^2 + \int_0^1 (u''(x))^2 dx \le 2M^2,$$

and

$$\|u_n\|_{W_1}^2 = (u_n(0))^2 + \int_0^1 (u'(x))^2 dx \le 2M^2.$$

Therefore, $\{u_n(x)\}_{n=1}^{\infty}$, $\{u_n'(x)\}_{n=1}^{\infty}$, $\{u_n''(x)\}_{n=1}^{\infty}$ and $\{u_n'''(x)\}_{n=1}^{\infty}$ are bounded sets in $W_1[0,1]$ respectively. This completes the proof. □

Lemma 2.5. *If $\|u_n - u\|_{W_5} \to 0$, $x_n \to x$, $(n \to \infty)$ and $f(x,y,z,w,v)$, $x \in [0,1]$, $y, z, w, v \in (-\infty, +\infty)$ is continuous with respect to x, y, z, w, v, then*

$$f(x_n, u_{n-1}(x_n), u_{n-1}'(x_n), u_{n-1}''(x_n), u_{n-1}'''(x_n)) \to f(x, u(x), u'(x), u''(x), u'''(x)) \quad \text{as } n \to \infty.$$

Proof. Since $\|u_n - u\|_{W_5} \to 0$, $(n \to \infty)$, by Lemma 2.2, we know that $u_n(x)$, $u_n'(x)$, $u_n''(x)$, $u_n'''(x)$ are convergent uniformly to $u(x)$, $u'(x)$, $u''(x)$, $u'''(x)$ respectively. □

3. An iterative method for the existence proof of BVPs (1.1)–(1.2)

Let $\{x_i\}_{i=1}^{\infty}$ be a dense set on [0, 1].
We construct an iterative sequence $u_n(x)$, putting

$$\begin{cases} \forall \text{ fixed } u_0(x) \in W_5[0,1], \\ u_n(x) = \sum_{i=1}^{n} A_i \bar{\psi}_i(x), \end{cases} \tag{3.1}$$

where

$$\begin{cases} A_1 = \beta_{11} f(x_1, u_0(x_1), u_0'(x_1), u_0''(x_1), u_0'''(x_1)), \\ A_2 = \sum_{k=1}^{2} \beta_{2k} f(x_k, u_{k-1}(x_k), u_{k-1}'(x_k), u_{k-1}''(x_k), u_{k-1}'''(x_k)), \\ \cdots \\ A_n = \sum_{k=1}^{n} \beta_{nk} f(x_k, u_{k-1}(x_k), u_{k-1}'(x_k), u_{k-1}''(x_k), u_{k-1}'''(x_k)). \end{cases} \tag{3.2}$$

Lemma 3.1. *$\{u_n\}_{n=1}^{\infty}$ in (3.1) is monotone increasing in the sense of $\|\cdot\|_{W_5}$.*

Proof. By Lemma 2.1, $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is a complete normal orthogonal system in $W_5[0, 1]$, hence we have

$$\|u_n\|_{W_5}^2 = \langle u_n(x), u_n(x)\rangle_{W_5} = \left\langle \sum_{i=1}^{n} A_i\overline{\psi}_i(x), \sum_{i=1}^{n} A_i\overline{\psi}_i(x) \right\rangle_{W_5} = \sum_{i=1}^{n}(A_i)^2.$$

Therefore, $\|u_n\|_{W_5}$ is monotone increasing. □

Lemma 3.2.

$$\mathbb{L}u_n(x_j) = f(x_j, u_{j-1}(x_j), u_{j-1}'(x_j), u_{j-1}''(x_j), u_{j-1}'''(x_j)), \quad j \le n. \tag{3.3}$$

Proof. Let $j \le n$,

$$\begin{aligned} \mathbb{L}u_n(x_j) &= \sum_{i=1}^{n} A_i \mathbb{L}\overline{\psi}_i(x_j) = \sum_{i=1}^{n} A_i \langle \mathbb{L}\overline{\psi}_i, \varphi_j\rangle_{W_1} \\ &= \sum_{i=1}^{n} A_i \langle \overline{\psi}_i, \mathbb{L}^*\varphi_j\rangle_{W_5} = \sum_{i=1}^{n} A_i \langle \overline{\psi}_i, \psi_j\rangle_{W_5} \end{aligned}$$

The orthonormality of $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ yields that

$$\begin{aligned} \sum_{l=1}^{j} \beta_{jl}\mathbb{L}u_n(x_l) &= \sum_{i=1}^{n} A_i \left\langle \overline{\psi}_i, \sum_{l=1}^{j}\beta_{jl}\psi_l \right\rangle_{W_5} = \sum_{i=1}^{n} A_i \langle \overline{\psi}_i, \overline{\psi}_j\rangle_{W_5} \\ &= A_j = \sum_{k=1}^{j} \beta_{jk} f(x_k, u_{k-1}(x_k), u_{k-1}'(x_k), u_{k-1}''(x_k), u_{k-1}'''(x_k)). \end{aligned}$$

When $j = 1$, then

$$\mathbb{L}u_n(x_1) = f(x_1, u_0(x_1), u_0'(x_1), u_0''(x_1), u_0'''(x_1)).$$

If $j = 2$, then

$$\beta_{21}\mathbb{L}u_n(x_1) + \beta_{22}\mathbb{L}u_n(x_2) = \beta_{21} f(x_1, u_0(x_1), u_0'(x_1), u_0''(x_1), u_0'''(x_1)) + \beta_{22} f(x_2, u_1(x_2), u_1'(x_2), u_1''(x_2), u_1'''(x_2)).$$

It is clear that

$$\mathbb{L}u_n(x_2) = f(x_2, u_1(x_2), u_1'(x_2), u_1''(x_2), u_1'''(x_2)).$$

Furthermore, it is easy to see by induction that

$$\mathbb{L}u_n(x_j) = f(x_j, u_{j-1}(x_j), u_{j-1}'(x_j), u_{j-1}''(x_j), u_{j-1}'''(x_j)). \quad \square$$

Theorem 3.1. *Suppose the following conditions are satisfied:*

(i) *$f(x, y, z, w, v) \in W_1[0, 1]$ for any $y = y(x)$, $z = z(x)$, $w = w(x)$ and $v = v(x) \in W_1[0, 1]$, where $x \in [0, 1]$, $y, z, w, v \in (-\infty, +\infty)$;*

(ii) $f(x, y, z, w, v), f_x(x, y, z, w, v), f_y(x, y, z, w, v), f_z(x, y, z, w, v), f_w(x, y, z, w, v)$ *and* $f_v(x, y, z, w, v)$ *are continuous bounded functions for* $x \in [0, 1], y, z, w, v \in (-\infty, +\infty)$. *Then* $\|u_n\|_{W_5}$ *is bounded.*

Proof. 1° Since $\{x_i\}_{i=1}^{\infty}$ is dense on interval $[0, 1]$, for any $x \in [0, 1]$, there exists a subsequence $\{x_{n_j}\}$, such that

$$x_{n_j} \to x, \quad \text{as } j \to \infty.$$

By $n_{j+1} - 1 \geq n_j$ and Lemma 3.2, we have

$$\mathbb{L}u_{n_{j+1}-1}(x_{n_j}) = f(x_{n_j}, u_{n_j-1}(x_{n_j}), u'_{n_j-1}(x_{n_j}), u''_{n_j-1}(x_{n_j}), u'''_{n_j-1}(x_{n_j})). \tag{3.4}$$

Boundedness of f implies that point range $\{\mathbb{L}u_{n_{j+1}-1}(x_{n_j})\}_{j=1}^{\infty}$ is bounded, there exists a subsequence of $\{\mathbb{L}u_{n_{j+1}-1}(x_{n_j})\}_{j=1}^{\infty}$, without loss of generality, denoted by $\{\mathbb{L}u_{n_{j+1}-1}(x_{n_j})\}_{j=1}^{\infty}$, such that $\mathbb{L}u_{n_{j+1}-1}(x_{n_j})$ is convergent. Then

$$\begin{aligned}
\lim_{n_j\to\infty} \mathbb{L}u_{n_{j+1}-1}(x_{n_j}) &= \lim_{n_j\to\infty} \sum_{i=1}^{n_{j+1}-1} A_i \mathbb{L}\overline{\psi}_i(x_{n_j}) = \sum_{i=1}^{\infty} A_i \mathbb{L}\overline{\psi}_i(x) \\
&= \lim_{n_j\to\infty} \sum_{i=1}^{n_{j+1}-1} A_i \mathbb{L}\overline{\psi}_i(x) = \lim_{n_j\to\infty} \mathbb{L}u_{n_{j+1}-1}(x).
\end{aligned} \tag{3.5}$$

Therefore, $\{\mathbb{L}u_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$ is uniformly bounded on $[0, 1]$, i.e. $\{u^{(4)}_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$ is uniformly bounded on $[0, 1]$.

2° Since $K_5(y, x)$ is the reproducing kernel in $W_5[0, 1]$, then

$$\begin{aligned}
u_{n_{j+1}-1}(x) &= \langle u_{n_{j+1}-1}(y), K_5(y, x)\rangle_{W_5} = u^{(4)}_{n_{j+1}-1}(0)\frac{\partial^4 K_5(0, x)}{\partial y^4} + \int_0^1 u^{(5)}_{n_{j+1}-1}(y)\frac{\partial^5 K_5(y, x)}{\partial y^5}\mathrm{d}y \\
&= u^{(4)}_{n_{j+1}-1}(0)\frac{\partial^4 K_5(0, x)}{\partial y^4} + u^{(4)}_{n_{j+1}-1}(1)\frac{\partial^5 K_5(1, x)}{\partial y^5} - u^{(4)}_{n_{j+1}-1}(0)\frac{\partial^5 K_5(0, x)}{\partial y^5} - \int_0^1 u^{(4)}_{n_{j+1}-1}(y)\frac{\partial^6 K_5(y, x)}{\partial y^6}\mathrm{d}y,
\end{aligned}$$

$$u'_{n_{j+1}-1}(x) = u^{(4)}_{n_{j+1}-1}(0)\frac{\partial^5 K_5(0, x)}{\partial y^4 \partial x} + u^{(4)}_{n_{j+1}-1}(1)\frac{\partial^6 K_5(1, x)}{\partial y^5 \partial x} - u^{(4)}_{n_{j+1}-1}(0)\frac{\partial^6 K_5(0, x)}{\partial y^5 \partial x} - \int_0^1 u^{(4)}_{n_{j+1}-1}(y)\frac{\partial^7 K_5(y, x)}{\partial y^6 \partial x}\mathrm{d}y.$$

Similarly,

$$u''_{n_{j+1}-1}(x) = u^{(4)}_{n_{j+1}-1}(0)\frac{\partial^6 K_5(0, x)}{\partial y^4 \partial x^2} + u^{(4)}_{n_{j+1}-1}(1)\frac{\partial^7 K_5(1, x)}{\partial y^5 \partial x^2} - u^{(4)}_{n_{j+1}-1}(0)\frac{\partial^7 K_5(0, x)}{\partial y^5 \partial x^2} - \int_0^1 u^{(4)}_{n_{j+1}-1}(y)\frac{\partial^8 K_5(y, x)}{\partial y^6 \partial x^2}\mathrm{d}y,$$

and

$$u'''_{n_{j+1}-1}(x) = u^{(4)}_{n_{j+1}-1}(0)\frac{\partial^7 K_5(0, x)}{\partial y^4 \partial x^3} + u^{(4)}_{n_{j+1}-1}(1)\frac{\partial^8 K_5(1, x)}{\partial y^5 \partial x^3} - u^{(4)}_{n_{j+1}-1}(0)\frac{\partial^8 K_5(0, x)}{\partial y^5 \partial x^3} - \int_0^1 u^{(4)}_{n_{j+1}-1}(y)\frac{\partial^9 K_5(y, x)}{\partial y^6 \partial x^3}\mathrm{d}y.$$

Applying Corollaries 2.1 and 2.2, we obtain that $\{u_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$, $\{u'_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$, $\{u''_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$ and $\{u'''_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$ are uniformly bounded on $[0, 1]$. In view of Lemma 2.4, $\{u_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$, $\{u'_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$, $\{u''_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$ and $\{u'''_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$ are compact sets in $C[0, 1]$, so there exist a subsequence of $\{u_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$ and $\bar{u}(x) \in C[0, 1]$, without loss of generality, denoted by $\{u_{n_{j+1}-1}(x)\}_{j=1}^{\infty}$, such that

$$\begin{cases}
u_{n_{j+1}-1}(x) \to \bar{u}(x), & (j \to \infty), \text{ (uniformly)}, \\
u'_{n_{j+1}-1}(x) \to \bar{u}'(x), & (j \to \infty), \text{ (uniformly)}, \\
u''_{n_{j+1}-1}(x) \to \bar{u}''(x), & (j \to \infty), \text{ (uniformly)}, \\
u'''_{n_{j+1}-1}(x) \to \bar{u}'''(x), & (j \to \infty), \text{ (uniformly)},
\end{cases}$$

Therefore, $\bar{u}(x)$, $\bar{u}'(x)$,$\bar{u}''(x)$ and $\bar{u}'''(x)$ are bounded on $[0, 1]$.

Taking limits in (3.4), it follows from (3.5) and Lemma 2.5,

$$\lim_{n_j\to\infty} \mathbb{L}u_{n_{j+1}-1}(x) = f(x, \bar{u}(x), \bar{u}'(x), \bar{u}''(x), \bar{u}'''(x)),$$

$$\int_0^x \lim_{n_j\to\infty} \mathbb{L}u_{n_{j+1}-1}(x)\mathrm{d}x = \int_0^x f(x, \bar{u}(x), \bar{u}'(x), \bar{u}''(x), \bar{u}'''(x))\mathrm{d}x,$$

$$\lim_{n_j\to\infty} (u'''_{n_{j+1}-1}(x) - u'''_{n_{j+1}-1}(0)) = \int_0^x f(x, \bar{u}(x), \bar{u}'(x), \bar{u}''(x), \bar{u}'''(x))\mathrm{d}x,$$

$$\bar{u}'''(x) - \bar{u}'''(0) = \int_0^x f(x, \bar{u}(x), \bar{u}'(x), \bar{u}''(x), \bar{u}'''(x))\mathrm{d}x,$$
$$\bar{u}^{(4)}(x) = f(x, \bar{u}(x), \bar{u}'(x), \bar{u}''(x), \bar{u}'''(x)).$$

From boundedness of f, we get $u^{(4)}(x)$ is bounded on $[0, 1]$.

$$\bar{u}^{(5)}(x) = f_x + f_y\bar{u}'(x) + f_z\bar{u}''(x) + f_w\bar{u}'''(x) + f_v\bar{u}^{(4)}(x).$$

Note that boundedness and continuity of f_x, f_y, f_z, f_w, f_v and $\bar{u}'(x), \bar{u}''(x), \bar{u}'''(x), \bar{u}^{(4)}(x)$, we can obtain that $\bar{u}^{(5)}(x)$ is bounded and continuous, therefore, $\bar{u}^{(5)}(x) \in L^2[0, 1]$ and $\bar{u}^{(4)}(x)$ is absolutely continuous in $[0, 1]$, i.e. $\bar{u}(x) \in W_5[0, 1]$. By Lemma 3.1, $\|u_{n_{j+1}-1}\|_{W_5} \leq \|u\|_{W_5}$. For any $n \in \mathbb{N}$, there exists $n_{j+1} - 1 \geq n$, such that $\|u_n\|_{W_5} \leq \|u_{n_{j+1}-1}\|_{W_5}$. □

Theorem 3.2. *Assume that the conditions of Theorem 3.1 are satisfied. Then $u_n(x)$ derived from the above iterative formula in (3.1) converges to the exact solution $u(x)$ of BVPs (1.1)–(1.2) in $W_5[0, 1]$ and*

$$u(x) = \sum_{i=1}^{\infty} A_i\bar{\psi}(x),$$

where A_i is given by (3.2).

Proof. (i) First of all, we will prove the convergence of $u_n(x)$.

From Lemma 3.1, we infer that

$$u_{n+1}(x) = u_n(x) + A_{n+1}\bar{\psi}_{n+1}(x), \qquad \|u_n\|^2_{W_5} = \sum_{i=1}^{n}(A_i)^2. \tag{3.6}$$

Due to the conditions that $\|u_n\|_{W_5}$ is bounded from Theorem 3.1 and $\|u_n\|_{W_5}$ is monotone increasing from Lemma 3.1, $\|u_n\|_{W_5}$ is convergent and there exists a constant c such that

$$\sum_{i=1}^{\infty}(A_i)^2 = c.$$

This implies that

$$\{A_i\}_{i=1}^{\infty} \in l^2.$$

If $m > n$, then

$$\|u_m - u_n\|^2_{W_5} = \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{n+1} - u_n\|^2_{W_5}.$$

In view of $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)$, it follows that

$$\|u_m - u_n\|^2_{W_5} = \|u_m - u_{m-1}\|^2_{W_5} + \|u_{m-1} - u_{m-2}\|^2_{W_5} \cdots + \|u_{n+1} - u_n\|^2_{W_5}.$$

Furthermore

$$\|u_m - u_{m-1}\|^2_{W_5} = (A_m)^2.$$

Consequently

$$\|u_m - u_n\|^2_{W_5} = \sum_{l=n+1}^{m}(A_l)^2 \to 0, \quad \text{as } n \to 0.$$

The completeness of $W_5[0, 1]$ shows that there exists a $u(x) \in W_5[0, 1]$ such that $u_n \to u$ as $n \to \infty$ in the sense of $\|\cdot\|_{W_5}$.

(ii) Second, we will prove that

$$\mathbb{L}u_n(x_j) = \mathbb{L}u(x_j), \quad j \leq n. \tag{3.7}$$

It follows that, on taking limits in (3.1),

$$u(x) = \sum_{i=1}^{\infty} A_i\bar{\psi}_i. \tag{3.8}$$

Therefore $u_n(x) = P_n u(x)$, where P_n is an orthogonal projector from $W_5[0, 1]$ to $\mathrm{Span}\{\psi_1, \psi_2, \ldots, \psi_n\}$.

$$\begin{aligned}\mathbb{L}u_n(x_j) &= \langle \mathbb{L}u_n, \varphi_j\rangle_{W_1} = \langle u_n, \mathbb{L}^*\varphi_j\rangle_{W_5} = \langle P_n u, \psi_j\rangle_{W_5} \\ &= \langle u, P_n\psi_j\rangle_{W_5} = \langle u, \psi_j\rangle_{W_5} = \langle \mathbb{L}u, \varphi_j\rangle_{W_1} = \mathbb{L}u(x_j).\end{aligned}$$

(iii) Finally, we will prove that $u(x)$ is the solution of BVPs (1.1)–(1.2).

From (ii) and Lemma 3.2, we have

$$\mathbb{L}u(x_j) = f(x_j, u_{j-1}(x_j), u'_{j-1}(x_j), u''_{j-1}(x_j), u'''_{j-1}(x_j)). \tag{3.9}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on interval $[0, 1]$, for any $x \in [0, 1]$, there exists a subsequence $\{x_{n_j}\}$, such that

$$x_{n_j} \to x, \quad \text{as } j \to \infty.$$

From (3.9), it is easy to see that

$$\mathbb{L}u(x_{n_j}) = f(x_{n_j}, u_{n_j-1}(x_{n_j}), u'_{n_j-1}(x_{n_j}), u''_{n_j-1}(x_{n_j}), u'''_{n_j-1}(x_{n_j})).$$

Hence, let $j \to \infty$, by Lemma 2.5 and the continuity of f, we have

$$\mathbb{L}u(x) = f(x, u(x), u'(x), u''(x), u'''(x)). \tag{3.10}$$

From (3.10), it follows that $u(x)$ satisfies Eq. (1.1).

Since $\bar{\psi}_i(x) \in W_5[0, 1]$, clearly, $u(x)$ satisfies the boundary conditions (1.2). That is, $u(x)$ is the solution of BVPs (1.1)–(1.2). □

From Lemma 2.2, we have the following the corollary.

Corollary 3.1. *Assume that the conditions of Theorem 3.2 hold, then $u_n(x)$ in 3.1 satisfies $\|u_n - u\|_{C^4[0,1]} \to 0$, $n \to \infty$, where $u(x)$ is the solution of BVPs (1.1)–(1.2).*

4. Conclusions

In summary, we use a constructive method to obtain the existence proof for solutions of a class of general fourth-order nonlinear boundary value problems in the reproducing kernel space. By the method, we present a sequence which is proved to converge to the exact solution uniformly. One of the advantages of the method is the conditions for determining solutions in BVPs (1.1)–(1.2) can be imposed on the reproducing kernel space. We can construct the reproducing kernel spaces satisfying diverse kinds of the conditions for determining solutions. Hence, it is concluded that the existence proof for diverse kinds of boundary conditions can be obtained in the similar manner.

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